E9 203 Compressed Sensing & Sparse SP.

03 May 2021.

Last time:

- Properties of RIC and ROC
- Thm. 6.8. $A \in \mathbb{C}^{m \times N}$, $2 \leq s \leq N$. Then,

  $m \geq \frac{cs}{\delta_s^2}$ if $N \geq Cm$ and $\delta_s \leq \delta_*$,

  where c, C and δ_* only dep. on each other.

  E.g. $c = \frac{1}{162}$, $C = 30$, $\delta_* = \frac{2}{3}$ is a valid choice.

Today:

- Implication of Thm. 6.8
- Analysis of BP via RIC.

Thm. 6.8 $\Rightarrow$ $\delta_s \geq \sqrt{\frac{cs}{m}}$ — ① $\left(m \geq \frac{cs}{\delta_s^2} \text{ is NECESSARY} \right)$

Now, $\exists$ matrices that satisfy the Welch bound.

Such matrices satisfy $\mu = \frac{c}{\sqrt{m}}$.

Prop. 6.2 (A w/ ℓ_2 normalized cols): $\delta_1 = 0$, $\delta_2 = \mu$,

$\delta_s \leq \mu_1(s-1) \leq (s-1)\mu$. $s \geq 2$.

This $\Rightarrow$ $\delta_s \leq (s-1)\mu \leq \frac{cs}{\sqrt{m}}$

$\delta_s \leq \frac{Cs}{\sqrt{m}}$ — ②

Since $\exists$ matrices that meet the Welch bound, $m \geq c' s^2$ is SUFFICIENT.

$$\sqrt{\frac{cs}{m}} \leq \delta_s \leq \frac{cs}{\sqrt{m}} \qquad \text{[Big gap!]}$$

For small δ_s, upper bound $\Rightarrow$ $m \geq c's^2$ sufficient
lower bound $\Rightarrow$ $m \geq c''s$ necessary.

Will see: (a) Random $A \in \mathbb{C}^{m \times N}$ satisfy $\delta_s \leq \delta$ with high prob. for any $\delta > 0$ provided

$$m \geq C \delta^{-2} s \log\left(\frac{eN}{s}\right).$$

(b) $\delta_s \leq \delta$ requires $m \geq c_\delta s \log\left(\frac{eN}{s}\right)$.

Thus, the ineq. $m \geq c's$ is optimal up to logarithmic scaling factors.

Analysis of Basis Pursuit

Lemma 6.10: Given $p, q > 0$, if $u \in \mathbb{C}^s$, $v \in \mathbb{C}^t$ satisfy $\max_{i \in [s]} |u_i| \leq \min_{j \in [t]} |v_j|$, then

$$\boxed{\|u\|_q \leq \frac{s^{1/q}}{t^{1/p}} \|v\|_p.}$$

(Special case: $p = 1$, $q = 2$, $t = s$ $\Rightarrow$

$\|u\|_2 \leq \frac{s^{1/2}}{s} \|v\|_1 = \frac{1}{\sqrt{s}} \|v\|_1$.)

Proof: $\frac{\|u\|_q}{s^{1/q}} = \left[\frac{1}{s} \sum_{i=1}^{s} |u_i|^q \right]^{1/q} \leq \max_{i \in [s]} |u_i|$

$\leq$

$\frac{\|v\|_p}{t^{1/p}} = \left[\frac{1}{t} \sum_{j=1}^{t} |v_j|^p \right]^{1/p} \geq \min_{j \in [t]} |v_j|$

$$\frac{\|v\|_p}{t^{1/p}} \geq \frac{\|u\|_q}{s^{1/q}} \Rightarrow \|u\|_q \leq \frac{s^{1/q}}{t^{1/p}} \|v\|_p. \quad \square$$

Thm. 6.9 Suppose the $(2s)^{th}$ RIC of $A \in \mathbb{C}^{m \times N}$ satisfies $\boxed{\delta_{2s} < \frac{1}{3}}$. Then every s-sparse $x \in \mathbb{C}^N$ is the unique soln. of

$$\min_{z \in \mathbb{C}^N} \|z\|_1 \quad \text{s.t. } Az = Ax \qquad (P_1)$$

Proof: (Recall Thm. 4.5: Given $A \in \mathbb{C}^{m \times N}$, every s-sparse $x \in \mathbb{C}^N$ is the unique soln of (P_1) iff A satisfies NSP(s), i.e., for any $S \subset [N]$, $|S| \leq s$, $\|v_S\|_1 < \|v_{\bar{S}}\|_1$ $\forall v \in \mathcal{N}(A) \setminus \{0\}$.)

Thus, it suffices t.s.t. $\|v_S\|_1 < \frac{1}{2} \|v\|_1$ $\forall v \in \mathcal{N}(A) \setminus \{0\}$.

(Appendix A: If x has at most s nonzero entries, $\|x\|_1 \leq \sqrt{s} \|x\|_2$.)

Let $\rho \triangleq \frac{2\delta_{2s}}{1-\delta_{2s}}$, and note that $\rho < 1$ whenever $\delta_{2s} < \frac{1}{3}$.

It suffices t.s.t. $\|v_S\|_2 \leq \frac{\rho}{2\sqrt{s}} \|v\|_1$ $\forall v \in \mathcal{N}(A)$,

$S \subset [N]$, $|S| = s$, $\rho \triangleq \frac{2\delta_{2s}}{1-\delta_{2s}}$.

Given $v \in \mathcal{N}(A)$, suffices to consider $S =: S_0$, where

S_0 = Idx. set containing the s largest abs. entries of v.

S_1 = " " " " " " " of v in S_0^c

S_2 = " — " — " of v in $(S_0 \cup S_1)^c$

$\vdots$

Note that $S_1, S_2 \ldots$ partition S_0^c, i.e., $S_0^c = S_1 \cup S_2 \cup \cdots$

Since $v \in \mathcal{N}(A)$, $Av_{S_0} = A(-v_{S_1} - v_{S_2} - \cdots)$

(RIC: $(1-\delta_s)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1+\delta_s)\|x\|_2^2$ $\forall$ s-sparse x)

$\Rightarrow \underbrace{\|v_{S_0}\|_2^2}_{\substack{s\text{-sparse} \\ \Rightarrow 2s\text{-sparse}}} \leq \frac{1}{(1-\delta_{2s})} \|Av_{S_0}\|_2^2 = \frac{1}{1-\delta_{2s}} \langle Av_{S_0}, A(-v_{S_1}) + A(-v_{S_2}) + \cdots \rangle$

$$= \frac{1}{1-\delta_{2s}} \sum_{k \geq 1} \langle Av_{S_0}, A(-v_{S_k}) \rangle \quad \text{— } (*)$$

(Prop. 6.3: $u, v \in \mathbb{C}^N$, $\|u\|_0 \leq s$, $\|v\|_0 \leq t$, $\text{supp}(u) \cap \text{supp}(v) = \phi$, then $|\langle Au, Av \rangle| \leq \delta_{s+t} \|u\|_2 \cdot \|v\|_2$)

Sub. " in ⊛, divide by $\|v_{S_0}\|_2 > 0$:

$$\|v_{S_0}\|_2 \leq \boxed{\frac{\delta_{2s}}{1-\delta_{2s}}} \sum_{k\geq 1} \|v_{S_k}\|_2 \qquad \rho/2$$

For any $k \geq 1$, the s absolute entries of v_{S_k} are $\leq$ all s absolute entries of $v_{S_{k-1}}$.

$\Rightarrow$ Using Lemma 6.10, $\|v_{S_k}\|_2 \leq \frac{1}{\sqrt{s}} \|v_{S_{k-1}}\|_1$

(largest $\leq$ smallest)

$$\Rightarrow \|v_{S_0}\|_2 \leq \frac{\rho}{2\sqrt{s}} \sum_{k\geq 1} \|v_{S_{k-1}}\|_1 \leq \frac{\rho}{2\sqrt{s}} \|v\|_1$$

as desired. ☒

Thm. 6.12 Suppose the $(2s)^{th}$ RIC of $A \in \mathbb{C}^{m\times N}$ satisfies

$$\delta_{2s} < \frac{4}{\sqrt{41}} \approx 0.6246.$$

Then, for any $x \in \mathbb{C}^N$ and $y \in \mathbb{C}^m$ with $\|Ax - y\|_2 \leq \eta$, a soln $x^\#$ of

$$\min_{z \in \mathbb{C}^N} \|z\|_1 \quad \text{s.t.} \quad \|Az - y\|_2 \leq \eta \qquad (P_{1,\eta})$$

approximates x with error

$$\|x - x^\#\|_1 \leq C\,\sigma_s(x)_1 + D\sqrt{s}\,\eta$$

$$\|x - x^\#\|_2 \leq \frac{C}{\sqrt{s}}\,\sigma_s(x)_1 + D\eta$$

where consts. $C, D > 0$ depend only on δ_{2s}.
